Non-asymptotic Analysis of Biased Stochastic Approximation Scheme

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Stochastic Approximation

- Objective: Find a stationary point of a smooth Lyapunov function \( V(\eta) \).
- SA scheme (Robbins and Monro, 1951): is a stochastic process,
\[
\eta_{n+1} = \eta_n - \gamma_n; H_n(X_{n+1}) \quad n \in \mathbb{N}
\]
where \( \eta \in \mathbb{R}^d \) is the nth state, \( \gamma > 0 \) is the step size.
- The drift term \( H_n(X_{n+1}) \) depends on an i.i.d. random element \( X_{n+1} \) and \( h(\eta) = \mathbb{E}[H_n(X_{n+1})|F_n] = SV(\eta) \),
where \( F_n = \sigma(\eta_1, X_{m:n}) \). In this case, SA is better known as the SGD method.

Biased SA Scheme

- The mean field is biased \( \Rightarrow \) gradient is sometimes difficult to compute...
- We have \( h(\eta_n) = SV(\eta_n) \) and for some \( c_0 \geq 0, c_1 > 0 \),
\[
c_0 + c_1 (\mathbb{V}(\eta_n)|h(\eta_n)|) \geq \mathbb{V}(h(\eta_n))^2, \quad \forall \eta \in \mathbb{H}
\]
- The drift term \( H_n(X_{n+1}) \) is not L.D.D., for example, in reinforcement learning, \( \eta \) controls the policy in a MDP & \( H_n(X_{n+1}) \) is computed from the MDP's state.
The random elements \( X_{n+1} \) form a state-dependent Markov chain:
\[
H_n(X_{n+1}) = \mathbb{P}(X_{n+1} = X_n | X_n, \eta_n, \mathcal{F}_n)
\]
where \( \mathbb{P} : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R} \) is Markov kernel with a unique stationary distribution \( \nu_0 \).
- In the latter case, the mean field is given by \( h(\eta) = \mathbb{E}[H_n(X_{n+1})|\mathcal{F}_n] \).
- Stopping criterion: fix any \( \varepsilon > 1 \), we stop the SA at a random iteration \( N \) with
\[
\mathbb{P}(N = n) = \left( \sum_{n=0}^{\infty} \nu_0(\eta_n)^{-1} \right) \nu_0(\eta_1), \quad \nu_0(\eta_1) \in [1, \ldots, n]
\]

Prior Work

- We focus on the non-asymptotic convergence analysis of SA scheme, where the relevant results are rare. Define:
\[
\eta_{n+1} = H_n(X_{n+1}) - h(\eta_n)
\]

Case 1: When \( \eta_{n+1} \) is Martingale difference \( \Rightarrow \|\eta_{n+1}\| = 0 \).
- Asymptotic analysis: (Robbins and Monro, 1951): Non-asymptotic analysis: (Ghadimi and Lan, 2013).

Case 2: When \( \eta_{n+1} \) is a state-controlled Markov noise
\[
(\eta_{n+1}|\mathcal{F}_n) = P_n H_n(X_{n+1}) - h(\eta_n) \neq 0
\]
- Asymptotic analysis: (Tadic and Doucet, 2017): Non-asymptotic analysis: (Sun et al., 2018, Durck et al., 2012, Ghadimi et al., 2018).

Analysis For Martingale Difference Case (1)

- Assumption: \( \mathbb{E}[k_n^2 | \mathcal{F}_n] = 0, \mathbb{E}[k_n^2 | \eta_n] \leq \sigma_2^2(\|h(\eta_n)\|^2) \). (e.g., when \( X_n \) is i.i.d. the SGD setting).

Theorem 1. Let \( \nu_0(\eta_1) \leq 2(2c_1(1+\sqrt{d_2})^{-1})^{-1} \) and \( \nu_0(\eta) = \mathbb{E}[V(\eta) - V(\eta)] \).
\[
\mathbb{E}[\nu_0(\eta_1)] \leq 2\sigma_2^2 \sum_{n=0}^{\infty} \gamma_n^2 \mathbb{E}[\|k_n\|_2^2] + 2c_0
\]
Set \( \eta_0 = (2(2c_1(1+\sqrt{d_2})^{-1})^{-1} = \mathbb{E}[(k_n)^2]) - C_0(c_0 + \log n) \). Remark: if \( h(\eta) = \mathbb{V}(\eta) \) (with \( c_0 = d_2 = 0 \)), it recovers (Ghadimi and Lan, 2013, Theorem 2.1).